ON VASSILIEV KNOT INVARIANTS INDUCED
FROM FINITE TYPE 3-MANIFOLD INVARIANTS

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ABSTRACT. We prove that the knot invariant induced by a $\mathbb{Z}$-homology 3-sphere invariant of order $\leq k$ in Ohtsuki's sense, where $k \geq 4$, is of order $\leq k - 2$. The method developed in our computation shows that there is no $\mathbb{Z}$-homology 3-sphere invariant of order 5.

1. INTRODUCTION

The theory of finite type knot invariants is by now well developed (see [V], [Gu], [B-L], [K] and [BN1]). It provides a unified approach to various quantum knot invariants (Jones, HOMFLY and Kauffman polynomials, etc.). Such an analogous unified approach to quantum 3-manifold invariants is not available at present.

In [O1], Ohtsuki reorganized quantum $SU(2)$ invariants $\tau_r(M)$ for a $\mathbb{Z}$-homology 3-sphere M into a formal power series

$$\tau(M) = \sum_{n=0}^{\infty} \lambda_n(M)(t-1)^n.$$

In order to find a characterization of the invariants $\lambda_n(M)$, Ohtsuki introduced the notion of finite type $\mathbb{Z}$-homology 3-sphere invariants. Precisely, let L be an algebraically split link with framings ± 1 on each component in a $\mathbb{Z}$-homology 3-sphere M and we will denote by M_L the $\mathbb{Z}$-homology 3-sphere obtained by surgery on L. Let λ be a $\mathbb{Z}$-homology 3-sphere invariant, then λ is said to be a *finite type invariant* of order $\leq k$ if the following sum vanishes for all choices of M and L such that $\#L > k$:

$$(1.1) \qquad \sum_{L' \subset L} (-1)^{\#L'} \lambda(M_{L'}) = 0.$$

Here $\#L$ is the number of components of the link L.

Let $\mathcal{O}_k$ be the vector space of all finite type invariants of $\mathbb{Z}$-homology 3-spheres of order $\leq k$. The main result of [O2] is that $\dim(\mathcal{O}_k) < \infty$. Moreover, it is proved that the "k-th difference" of a finite type $\mathbb{Z}$-homology 3-sphere invariant of order k is determined by its "weight" on the set of graphs of k edges such that (1) the valence at each vertex is either 1 or 3 and (2) every edge connects two distinct vertices.

In [G], among other things, Garoufalidis proved that $\mathcal{O}_4 = \mathcal{O}_3$. He also observed that every $\lambda \in \mathcal{O}_k$ determines a finite type knot invariant v_λ of order $\leq k - 1$ by

$$(1.2) \qquad\qquad v_\lambda(K) = \lambda(S^3) - \lambda(S_K^3)$$

for every knot K in S^3 with $+1$ framing. A way of calculating the weight system of v_λ was also described in [G]. Garoufalidis suggested that it might be interesting to study these induced knot invariants, although it was also observed that the presence of Kirby moves may restrict the induced knot invariants considerably.

In this paper, we examine closely Ohtsuki's proof of the finiteness of $\dim(\mathcal{O}_k)$. This leads to one of our main results showing that the order of an induced knot invariant is, in most cases, actually one less than what it was supposed to be *a priori*.

Theorem 1.1. *Let v_λ be the knot invariant induced by $\lambda \in \mathcal{O}_k$ with $k \geq 4$. Then v_λ is of order at most $k - 2$.*

Our method also leads to the determination of $\mathbb{Z}$-homology 3-sphere invariants of order ≤ 5.

Theorem 1.2. $\mathcal{O}_5 = \mathcal{O}_4$.

In [R1], [R2] Rozansky has shown, using physical predictions of the asymptotic behavior of Witten's Chern-Simons path integral over trivial connections, that Ohtsuki's invariant λ_n is of order $3n$ and he further conjectured that $\mathcal{O}_{3n} = \mathcal{O}_{3n+1} = \mathcal{O}_{3n+2}$. The results of this paper certainly point in this direction.

Let λ_C be the Casson invariant and 1 be the constant invariant whose value on each $\mathbb{Z}$-homology 3-sphere is 1. It is known that $\{1, \lambda_\mathrm{C}\}$ is a basis for $\mathcal{O}_3$ (see [O2] and [G]). Let v_C be the knot invariant induced by λ_C. It is known that v_C is of order 2. To illustrate Theorem 1.1, we may consider the order 6 $\mathbb{Z}$-homology 3-sphere invariant λ_C^2 whose induced knot invariant v_C^2 is of order 4. Probably also motivated by this example, Garoufalidis asked whether the knot invariant induced by a $\mathbb{Z}$-homology 3-sphere invariant of order $\leq 3n$ is actually of order $\leq 2n$. The answer to this question remains open.

Remark. Since the submission of this paper, much progress has been made on the structure of the filtered space $\mathcal{O} = \bigcup \mathcal{O}_k$. For example, it is now known that $\mathcal{O}_{3n} = \mathcal{O}_{3n+1} = \mathcal{O}_{3n+2}$ (Garoufalidis-Levine) and that $\lambda \in \mathcal{O}_{3n}$ implies that v_λ is of order at most $2n$ (Habegger and Garoufalidis-Levine). We will refer the reader to [Li] for a survey of, as well as references to, these and other developments. Nevertheless, we hope that this paper will still provide an interesting and illuminating piece in the puzzle assembled rapidly in the last year through the efforts of many people.

Acknowledgment. We appreciate greatly the referee's patience and persistence on the accuracy of the exposition.

2. Definitions and basic properties.

In this section we will first recall a few definitions and lemmas of Ohtsuki [O2]. Then we will give a proof of Ohtsuki's finiteness theorem using the notion of surgery classification of links in S^3. Our proof here will have the same essential ingredients as in Ohtsuki's proof. But we do think that our exposition is clearer than that of Ohtsuki's. We single out a key step in our argument, the proof of Theorem 2.2, in
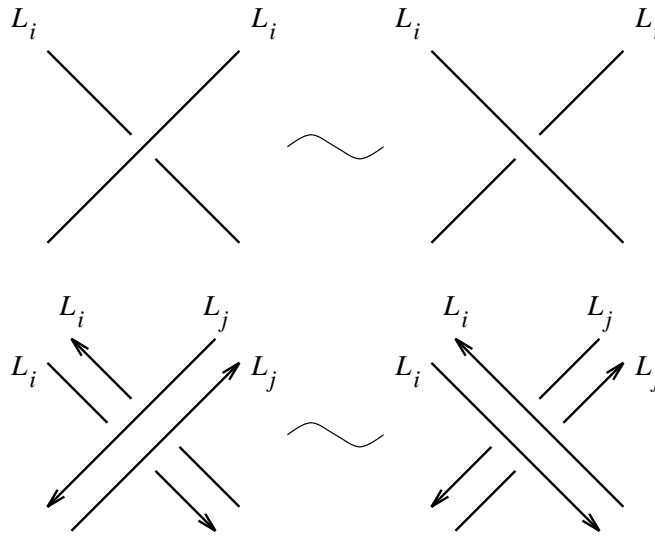
FIGURE 2.1. Local moves on ASL's.

the next section since the same method will be used in the proof of Theorem 1.1 in the last section.

An *algebraically split link* (ASL) L in a $\mathbb{Z}$-homology 3-sphere M is a link with unoriented and unordered components such that all of its linking numbers are zero. Henceforth, all 3-manifolds will be $\mathbb{Z}$-homology 3-spheres and all links will be ± 1-framed ASL's.

Let $\mathcal{I}$ be the set of homeomorphism classes of $\mathbb{Z}$-homology 3-spheres. Let $\mathcal{O}$ be the vector space over $\mathbb{C}$ with $\mathcal{I}$ as a basis. Define an element (M, L) in $\mathcal{O}$ by

$$(M, L) = \sum_{L' \subset L} (-1)^{\# L'} M_{L'}$$

where L' runs through all sublinks (including the empty sublink) of L. Let $\mathcal{O}_k$ be the subspace of $\mathcal{O}$ spanned by (M, L) for all $\mathbb{Z}$-homology 3-spheres M and all ± 1-framed links L such that $\# L = k + 1$. The space $\mathcal{O}$ has a natural stratification:

$$\mathcal{O} = \mathcal{O}_{-1} \supset \mathcal{O}_0 \supset \mathcal{O}_1 \supset \cdots \supset \mathcal{O}_{k-1} \supset \mathcal{O}_k \supset \cdots.$$

For $A, B \in \mathcal{O}_{k-1}$, we denote $A \sim B$ if $A - B \in \mathcal{O}_k$. We have the following basic lemmas from [O2].

Lemma 2.1. *For any $(M, L) \in \mathcal{O}_{k-1}$, there is a link J in S^3 with $\# J = k$ such that $(M, L) \sim (S^3, J)$.*

Lemma 2.2. *$(S^3, L) \sim (S^3, J)$ if J can be obtained from L by either of the local moves shown in Figure 2.1 where L_i and L_j may be different components of L. In the second local move, we may orient components of L in any favorable way.*

Lemma 2.3. *Let L and J be the same link in S^3 with different ± 1 framings. Let $s(L)$ be the product of framings of L. Then $s(L)(S^3, L) \sim s(J)(S^3, J)$.*

From these lemmas, we may think of $\mathcal{O}_{k-1}/\mathcal{O}_k$ as spanned by equivalence classes of ASL's in S^3 with k components under the equivalence relations depicted in

Figure 2.1, where an ASL L is regarded as an element $(S^3, L) \in \mathcal{O}_{k-1}$ with the understanding that the framings of L are all $+1$. Now it seems to be the right moment to recall the notion of surgery classification of links in S^3 here.

Definition 2.1. A *surgery modification* on an ASL L in S^3 is first to have a disk B in S^3 intersecting L only in its interior with zero algebraic intersection number. Then, perform a ± 1 surgery on ∂B which will change L to another ASL J in S^3. Two ASL's in S^3 are *surgery equivalent* if one of them can be changed to another by a finite sequence of surgery modifications.

Notice that Figure 2.1 also serves as an illustration of a surgery modification. Thus, we have

Lemma 2.4. *If L and J are surgery equivalent ASL's with k components, then $(S^3, L) \sim (S^3, J)$.*

To arrive at the finiteness of $\dim(\mathcal{O}_{k-1}/\mathcal{O}_k)$, we recall the following theorem of Levine [L].

Theorem 2.1. *Surgery equivalence classes of ASL's with oriented and ordered components are classified by Milnor's triple μ-invariants.*

We may now think of a surgery equivalence class of ASL's with k components as a collection of integers $\{\mu(i_1 i_2 i_3)\}$ where i_1, i_2, i_3 are distinct indices among $1, \ldots, k$. They satisfy the following relations:

$$\begin{cases} \mu(i_1 i_2 i_3) = \mu(i_2 i_3 i_1) = \mu(i_3 i_1 i_2), \\ \mu(i_1 i_2 i_3) = -\mu(i_2 i_1 i_3). \end{cases}$$

Taking advantage of being in the vector space $\mathcal{O}_{k-1}/\mathcal{O}_k$, each collection of integers $\{\mu(i_1 i_2 i_3)\}$ will be further simplified in the following theorem. This theorem will imply the finiteness of $\dim(\mathcal{O}_{k-1}/\mathcal{O}_k)$.

Theorem 2.2. *In $\mathcal{O}_{k-1}/\mathcal{O}_k$, an ASL representing $\{\mu(i_1 i_2 i_3)\}$ can be expressed as a linear combination of ASL's whose collections of triple μ-invariants satisfy the following conditions:*

1. *each $\mu(i_1 i_2 i_3)$ is either 0 or 1 for certain fixed cyclic order of indices;*
2. *each index i appears in at most two non-zero $\mu(i_1 i_2 i_3)$'s.*

Now collections $\{\mu(i_1 i_2 i_3)\}$ of 0's and 1's, where i_1, i_2, i_3 are distinct indices among $1, \ldots, k$ with a fixed cyclic ordering are in 1-1 correspondence with graphs with k ordered edges such that

1. each vertex is of valence either 1 or 3;
2. each edge connects two distinct vertices; and
3. incident edges of each valance 3 vertex come equipped with a cyclic order.

Furthermore, we may forget the ordering of edges of such a graph since the components of ASL's in $\mathcal{O}_{k-1}/\mathcal{O}_k$ need not to be ordered. Thus, we see that $\mathcal{O}_{k-1}/\mathcal{O}_k$ is spanned by ASL's in 1-1 correspondence with graphs having k edges and satisfying the properties (1), (2) and (3) above. In particular, we have the following corollary.

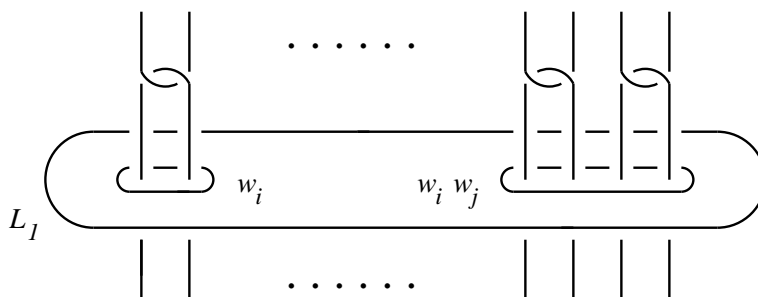**Corollary 2.1.** $\dim(\mathcal{O}_{k-1}/\mathcal{O}_k) < \infty$.

FIGURE 3.1. The standard picture of a component in an ASL.

3. THE PROOF OF THEOREM 2.2

We start with an ASL L in S^3 representing $\{\mu(i_1 i_2 i_3)\}$. We may assume that each component of L is unknotted. Notice that modifying one component of L will not affect the triple μ-invariants not involving the index of that component. This allows us to "localize" a component of L in the following way.

Pick a component, say L_1, of L. It bounds a disk B. Other components of L will intersect B in its interior only and the algebraic intersection numbers are all zero. Thus, we may pair the intersection points so that each pair consisting of two points lying on the same component of L, one with positive intersection number and the other with negative intersection number. On the disk B, we draw small circles $w_1, w_2, \ldots, w_n$ such that

1. each of them surrounds a pair of intersection points;
2. they are all disjoint; and
3. they are not nested with each other.

We may further assume that each small circle w_i represents a simple commutator $[m_{i_1}, m_{i_2}]$, where m_{i_1} and m_{i_2} are meridians of the i_1-th and i_2-th components, respectively, with $i_1 \neq i_2$. To see this, notice that we may recover the link L by starting with an unlink and performing "finger moves" between different components. The components corresponding to L_1 can remain untouched, but the fingers will run through the disk it bounds (which is B) many times. We have one small circle w_i surrounding each intersection of the fingers with B.

Abusing the notation, we will denote by the product $w_1 w_2 \cdots w_n$ the original link L, by w_i the link obtained from replacing the component L_1 by w_i, and by $w_i w_j$ the link obtained from replacing the component L_1 by a "nice" band-sum of w_i and w_j, with i, j distinct, on the disk B away from other small circles. See Figure 3.1.

Lemma 3.1. *In $\mathcal{O}_{k-1}/\mathcal{O}_k$, we have*

$$(3.1) \qquad w_1 w_2 \cdots w_n = \sum_{1 \leq i < j \leq n} w_i w_j - (n-2) \sum_{1 \leq i \leq n} w_i.$$

Proof. In Figures 3.2 and 3.3, we see local pictures of $+1$-framed ASL's with k components. Two basic relations in $\mathcal{O}_{k-1}/\mathcal{O}_k$, which are consequences of Lemma 2.2, are shown in Figure 3.2. The lemma is then proved inductively following Figure 3.3.
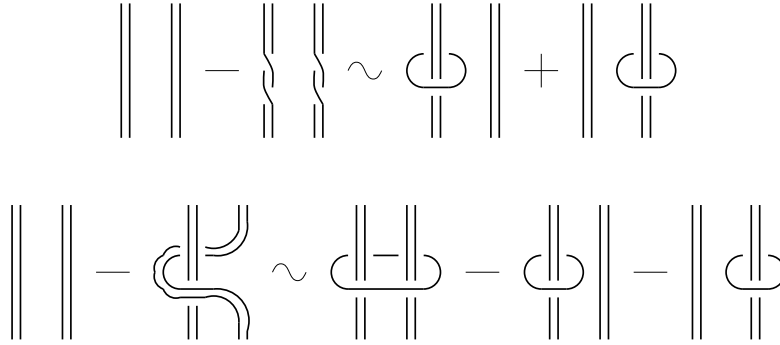
FIGURE 3.2. The basic relations in $\mathcal{O}_{k-1}/\mathcal{O}_k$.
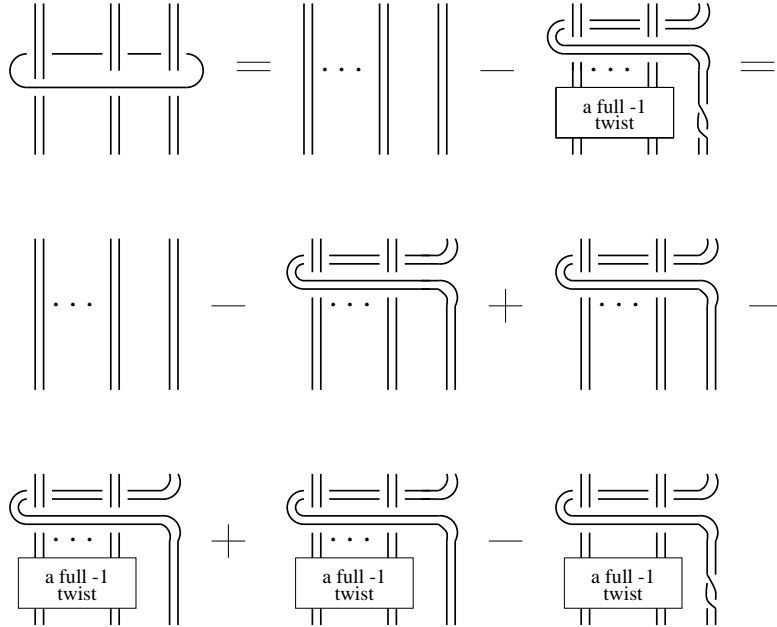

FIGURE 3.3. Prove Lemma 3.1 inductively.

There are three differences in the last expression in Figure 3.3. The first difference is, using the second basic relation $n-1$ times,

$$\sum_{1 \le i \le n-1} w_i w_n - \sum_{1 \le i \le n-1} w_i - (n-1)w_n.$$

The second difference is the ASL obtained by replacing the component L_1 with $w_1 \cdots w_{n-1}$, which equals to, inductively,

$$\sum_{1 \le i < j \le n-1} w_i w_j - (n-3) \sum_{1 \le i \le n-1} w_i.$$

Finally, the third difference is simply w_n. Putting these three differences together, we get the conclusion of Lemma 3.1. □
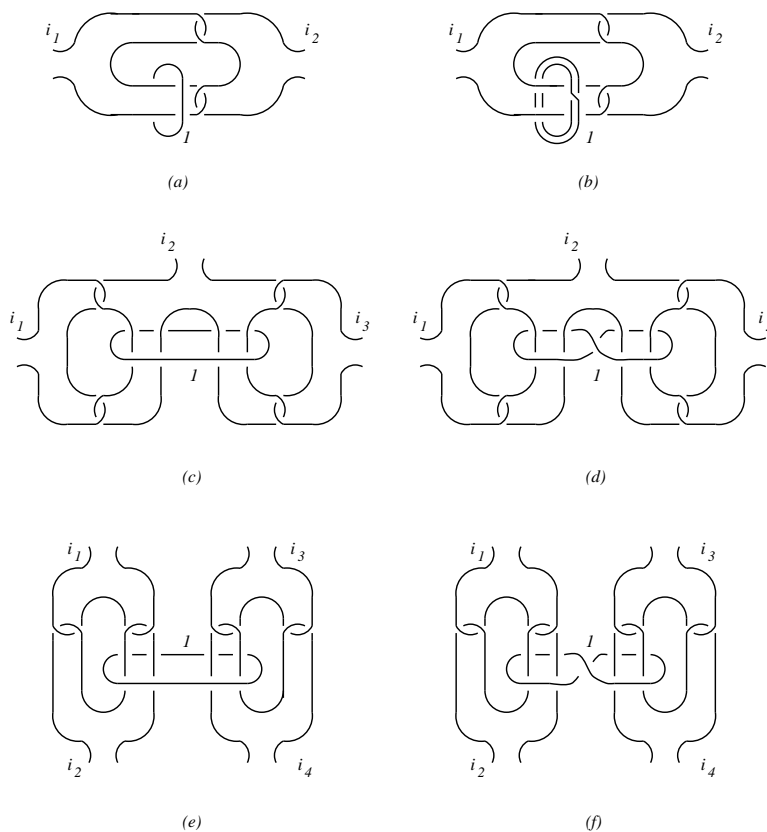
FIGURE 3.4. Localization of a component.

By Lemma 3.1, we only need to understand to what extent the component L_1 in each link w_i or $w_i w_j$, $i \neq j$, is "localized".

For the link w_i, there is only one triple μ-invariant involving the index 1: $\mu(1 i_1 i_2) = \pm 1$. Locally, it can be realized as in Figure 3.4 (a).

For the link $w_i w_j$, $i \neq j$, we need to consider several cases separately.

Case 1: No non-zero triple μ-invariants involving the index 1. In this case, $w_i w_j \equiv 1$ modulo commutators of weight ≥ 3. So $w_i w_j = 0$ in $\mathcal{O}_{k-1}/\mathcal{O}_k$.

Case 2: Only one non-zero triple μ-invariants involving the index 1: $\mu(1 i_1 i_2) = \pm 2$. Locally, it can be realized as in Figure 3.4 (b).

Case 3: Two non-zero triple μ-invariants involving the index 1:

$$\mu(1 i_1 i_2) \mu(1 i_2 i_3) = \pm 1$$

with i_1, i_2, i_3 distinct. Locally, they can be realized as in Figure 3.4 (c) and (d).

Case 4: Two non-zero triple μ-invariants involving the index 1:

$$\mu(1 i_1 i_2) \mu(1 i_3 i_4) = \pm 1$$

with i_1, i_2, i_3, i_4 distinct. Locally, they can be realized as in Figure 3.4 (e) and (f).

For the link in Figure 3.4 (a), we may assume that $\mu(1 i_1 i_2) = 1$ since it doesn't matter how the first component is oriented in $\mathcal{O}_{k-1}/\mathcal{O}_k$. Similarly, in Figure 3.4
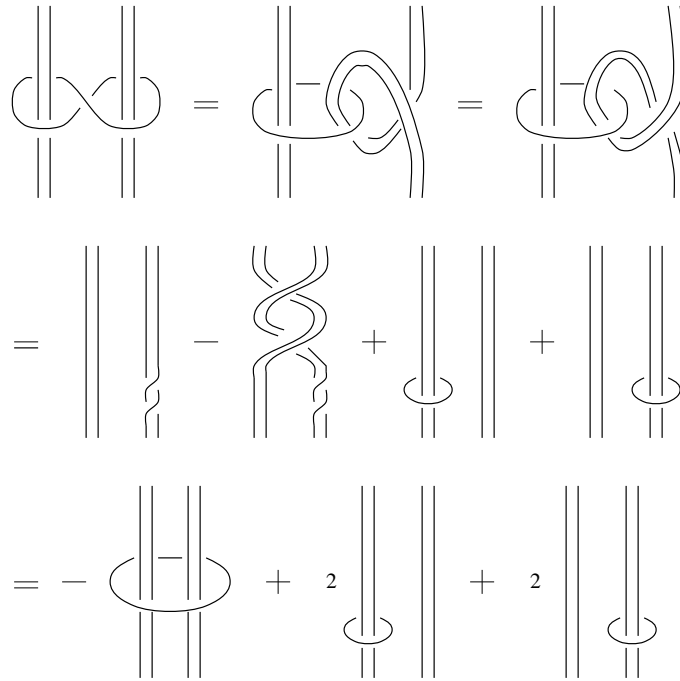
FIGURE 3.5. From a half twisted circle to untwisted ones.

(c) and (e), we may assume that the triple μ-invariants involving the index 1 all equal to 1.

Finally, it can be seen that the links in Figure 3.4 (b), (d) and (f), viewed as elements of $\mathcal{O}_{k-1}/\mathcal{O}_k$, can be expressed as linear combinations of links of the form shown in Figure 3.4 (a), (c) and (e). We refer the reader to Figure 3.5, where we see that a negatively half twisted circle can be reduced to a linear combination of the same link with several different untwisted circles respectively. The first and second equalities in Figure 3.5 use the fact that we are in $\mathcal{O}_{k-1}/\mathcal{O}_k$ (there will be some addtional twists on the kinky bands otherewise). The third equality uses the first relation in Figure 3.2. The last equality uses the second relation in Figure 3.2 and Lemma 2.3 twice.

Thus we have achieved those two properties in Theorem 2.2 for triple μ-invariants involving the index 1.

Once we have localized the first component, we proceed to localize the second component. If you look at the process of localizing a component, you will find out that the localization of the second component will affect $\mu(1i_1i_2)$ involving the index 2 only by possibly changing a non-zero one to zero. Keep localizing each component and this completes the proof of Theorem 2.2.

To simplify the exposition, by *simple graphs*, we will mean graphs which satisfy

1. the valence at each vertex is either 1 or 3;
2. every edge connects two distinct vertices;
3. incident edges at a valence 3 vertex have a cyclic order.

Remark. In $\mathcal{O}_{k-1}/\mathcal{O}_k$, it is obvious that changing the cyclic order of incident edges at a valence 3 vertex will only change a simple graph by a sign. So we may ignore the orders at valence 3 vertices for simplicity.

4. INDUCED KNOT INVARIANTS: THE PROOF OF THEOREM 1.1 AND 1.2

Let λ be a $\mathbb{Z}$-homology 3-sphere invariant. Let L be an ASL in S^3 with $+1$ framings. We define

$$(4.1) \qquad \psi_\lambda(L) = \sum_{L' \subset L} (-1)^{\#L'} \lambda(S_{L'}^3).$$

Lemma 4.1. *A $\mathbb{Z}$-homology 3-sphere invariant λ is of order $\leq k$ iff $\psi_\lambda(L) = 0$ for every ASL L in S^3 with $\#L > k$.*

Proof. This comes from the definition (1.1) and Lemma 2.1 as well as Lemma 2.3. $\square$

Analogous to the theory of finite type knot invariants, we may think of ASL's in S^3 as "singular $\mathbb{Z}$-homology 3-spheres" and simple graphs as "chord diagrams". For a finite type $\mathbb{Z}$-homology 3-sphere invariant λ, ψ_λ may then be thought of as the "derivative" of λ. The following two lemmas have their counterparts in the theory of finite type knot invariants.

Lemma 4.2. *Let λ be a $\mathbb{Z}$-homology 3-sphere invariant of order $\leq k$. Then λ is determined by ψ_λ on ASL's in S^3 with $\leq k$ components.*

Proof. Let $M = S_L^3$ for a ± 1-framed ASL L in S^3. We prove the lemma inductively on the number of components of L.

If $L = \emptyset$, then $M = S^3 = (S^3, \emptyset)$. So the lemma holds. Now assume that $\#L = n$. Then,

$$\lambda(S_L^3) = (-1)^n \left(\psi_\lambda(L) - \sum_{L' \subset L, L' \neq L} (-1)^{\#L'} \lambda(S_{L'}^3) \right).$$

Inductively, $\lambda(S_{L'}^3)$ for $L' \subset L$, $L' \neq L$ is determined by ψ_λ on ASL's in S^3 with $\leq k$ components. Since $\psi_\lambda(L) = 0$ if $n > k$, we see that $\lambda(S_L^3)$ is also determined by ψ_λ on ASL's in S^3 with $\leq k$ components. $\square$

Lemma 4.3. *Let L be an ASL in S^3 and O be an unknot separated from L, then $\psi_\lambda(L \cup O) = 0$.*

Proof. We have

$$\psi_\lambda(L \cup O) = \sum_{L' \subset L, O \notin L'} (-1)^{\#L'} \lambda(S_{L'}^3) + \sum_{L' \subset L, O \in L'} (-1)^{\#L'} \lambda(S_{L'}^3)$$

$$= \sum_{L' \subset L, O \notin L'} (-1)^{\#L'} \lambda(S_{L'}^3) + \sum_{L' \subset L, O \notin L'} (-1)^{\#L'+1} \lambda(S_{L' \cup O}^3).$$

Notice that $S_{L' \cup O}^3 = S_{L'}^3$ for any sublink L' of L such that $O \notin L'$. Hence we have

$$\psi_\lambda(L \cup O) = \sum_{L' \subset L, O \notin L'} (-1)^{\#L'} \lambda(S_{L'}^3) - \sum_{L' \subset L, O \notin L'} (-1)^{\#L'} \lambda(S_{L'}^3) = 0.$$
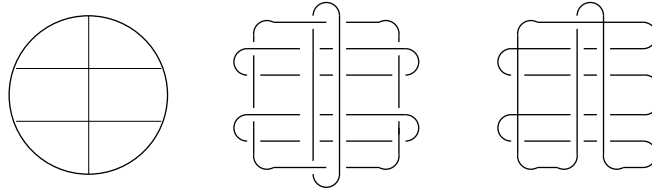
This finishes the proof. $\square$

FIGURE 4.1. A chord diagram, its associated ASL and a singular knot respecting it.

Restricting to the case $\#L = 1$, ψ_λ becomes a knot invariant. We emphasize this situation by having a separate notation: For a knot K in S^3, frame it by $+1$ and denote

(4.2) $$v_\lambda(K) = \lambda(S^3) - \lambda(S_K^3).$$

Notice that this definition of v_λ differs slightly from the definition, which is given in (1.2), used in the statement of Theorem 1.1. Obviously, the conclusions of Theorem 1.1 under these two definitions are equivalent. So we may use the current definition in the proof of Theorem 1.1.

Lemma 4.4. *If λ is a $\mathbb{Z}$-homology 3-sphere invariant of order $\leq k$, v_λ is a knot invariant of order $\leq k - 1$.*

Proof. This follows from Ohtsuki's first local move in Figure 2.1. For the evaluation of v_λ on a knot with k double points is equal to $\psi_\lambda(L)$ for a ± 1-framed ASL L in S^3 with $k + 1$ components, which is zero by definition of λ. □

Given any chord diagram C with $k - 1$ chords, one may obtain an ASL with k components in S^3 in the way depicted in Figure 4.1. We will first replace each chord by a thin untwisted band intersecting in its interior the outer circle at the end points of the original chord. These bands are disjoint from each other. We then take the boundaries of these bands together with the outer circle to form a ASL with k components. It is a well-defined element in $\mathcal{O}_{k-1}/\mathcal{O}_k$ by Ohtsuki's second local move. We denote by $L(C)$ this ASL associated with a chord diagram C. The following lemma was observed by Garoufalidis [G] (with a sign correction noticed by the referee of this paper).

Lemma 4.5. *The weight system for v_λ is given by*

$$v_\lambda(C) = (-1)^{k-1}\psi_\lambda(L(C))$$

for every chord diagram C with $k - 1$ chords.

Proof. Choose a "good" immersion for C (which is a singular knot respecting C). This is to be done in the following way [B-L]. Pick a point on the outer circle and follow it counterclockwise. When you come to a chord, follow the chord until it is about to meet the other edge of the outer circle, go underneath it and turn back immediately to create a double point with the outer circle, then follow the chord back to where it began, and continue around the outer circle, until you have returned to the starting point. See Figure 4.1. Notice now that the immersion can always be chosen in the following way. When each double point is resolved positively, a loop is created that can be isotoped back to the outer circle. The net effect is as if that chord in the chord diagram being erased. Further, when
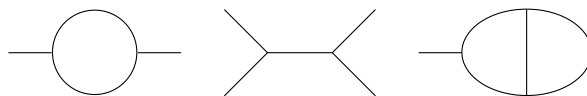
FIGURE 4.2. The bubble, the switch and the lollipop.

each double point is resolved negatively, the twist created in the outer circle is a negative one. The same effect can be certainly achieved by either performing or not performing (-1)-surgery on the component of $L(C)$ corresponding to that chord. Let L_0 be the outer circle of C thought of as a component of $L(C)$. Then

$$
\begin{aligned}
v_\lambda(C) &= \sum_{L' \subset L(C) - L_0} (-1)^{\#L'} \lambda(S^3_{L'}) - \sum_{L' \subset L(C) - L_0} (-1)^{\#L'} \lambda(S^3_{L' \cup L_0}) \\
&= \sum_{L_0 \notin L' \subset L(C)} (-1)^{\#L'} \lambda(S^3_{L'}) + \sum_{L_0 \in L' \subset L(C)} (-1)^{\#L'} \lambda(S^3_{L'}) \\
&= \sum_{L' \subset L(C)} (-1)^{\#L'} \lambda(S^3_{L'}) \\
&= (-1)^{k-1} \psi_\lambda(L(C)).
\end{aligned}
$$

where all the sublinks $L' \subset L(C) - L_0$ are -1-framed. The last equality follows from Lemma 2.3. □

We now use Lemma 3.1 to calculate the weight system for v_λ. Notice that the outer circle, call it L_0, in the ASL $L(C)$ associated with a chord diagram C is quite special. If we delete L_0, $L(C)$ will fall apart into an unlink. Therefore, L_0 represents an element in the free group $\pi_1(S^3 \setminus (L(C) - L_0))$ with $k - 1$ generators. Moreover, we can write out this element as a word in generators easily as follows. First, arbitrarily number the chords by $1, \ldots, k - 1$, and orient them. For each chord i, place the symbol i on the outgoing endpoint of the chord and $\bar{i}$ on the incoming endpoint. Pick a base point on the outer circle and read off the symbols encountered when traversing the circle counterclockwise. This is the word we are looking for. This word can be expressed as a product of simple commutators of the meridians of $L(C) - L_0$ modulo commutators of weight ≥ 3. If the number of simple commutators is > 2, we may use Lemma 3.1 to decompose this product of simple commutators into a sum of products of at most two simple commutators. Since there are no non-zero triple μ-invariants other than these involving the index 0 (of the component $L - L_0$), such a decomposition suffices to express $L(C)$ as a sum of simple graphs.

As an example, we first reprove Garoufalidis' result in the following lemma.

Lemma 4.6. $\mathcal{O}_4 = \mathcal{O}_3$.

Proof. There is only one simple graph of 4 edges, "the bubble" (see Figure 4.2), which is possibly non-zero in $\mathcal{O}_3/\mathcal{O}_4$. Suppose there is a $\mathbb{Z}$-homology 3-sphere invariant λ of order ≤ 4 and let us consider the induced knot invariant of order ≤ 3. There are two chord diagrams C_1 and C_2 with 3 chords which may have non-zero weights for an invariant of order 3. These are shown in Figure 4.3. The 4-term relation in the theory of finite type knot invariants implies that $v_\lambda(C_2) = 2v_\lambda(C_1)$. Thus, we must have
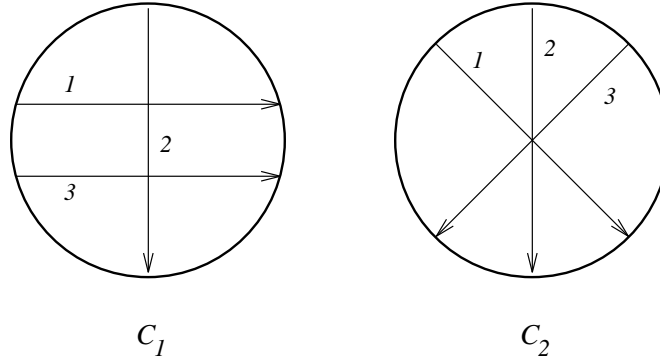
$$
\psi_\lambda(L(C_2)) = 2\psi_\lambda(L(C_1)).
$$

FIGURE 4.3. Chord diagrams with 3 chords.

Now consider their respective ASL's. It is not to hard to see that $L(C_1)$ can be decomposed directly as a simple graph, the bubble. Let us now decompose $L(C_2)$ into simple graphs. We label the chords of C_2 by 1,2,3 and obtain the following word

$$123\bar{1}\bar{2}\bar{3} = [1, 23][2, 3] \sim [1, 2][1, 3][2, 3]$$

By Lemma 3.1, we can decompose this word as

$$[1, 2][1, 3] + [1, 2][2, 3] + [1, 3][2, 3] - [1, 2] - [1, 3] - [2, 3].$$

The simple graphs corresponding to the last three words are zero in $\mathcal{O}_3/\mathcal{O}_4$ since they all have an isolated edge. The simple graphs corresponding to the first three words are all equal to the bubble. Thus

$$L(C_2) = 3L(C_1)$$

in $\mathcal{O}_3/\mathcal{O}_4$. This is possible only when $\lambda = 0$ on the bubble. Therefore, λ is of order ≤ 3. □

Proof of Theorems 1.1 and 1.2. For $k = 4$, the proof of Theorem 1.1 is given by Lemma 4.6. We jump to the case $k > 5$ first. Let $\lambda \in \mathcal{O}_k$ for $k > 5$. Let us evaluate the weight system of v_λ. For each chord diagram C with $k - 1 > 4$ chords, decompose the outer circle L_0 in $L(C)$ using Lemma 3.1 into a sum of products of at most two simple commutators. Since there are more than four chords, there will be at least one isolated edge in each of the simple graphs in the decomposition coming from Lemma 3.1 and $v_\lambda(C) = \lambda(L(C)) = 0$.

Finally, we come to the proof of Theorem 1.1 in the case when $k = 5$. In this case, there are two simple graph with 5 edges, "the switch" and "the lollipop", in $\mathcal{O}_4/\mathcal{O}_5$. See Figure 4.2. It is easy to see that the lollipop has to vanish in $\mathcal{O}_4/\mathcal{O}_5$: We may twist the tail to change the order of incident edges at a vertex, but that twist can be undone by twisting the body of the lollipop. So we only need to check the switch.

Unlike in the proof of Lemma 4.6, the induced weight system on chord diagrams with 4 chords is actually consistent as a one-dimensional vector space, whose basis element is the switch.

Nevertheless, as we will see, the local moves in Figure 2.1 put more constraints on induced knot invariants than merely the 4-term relation. Let us consider the
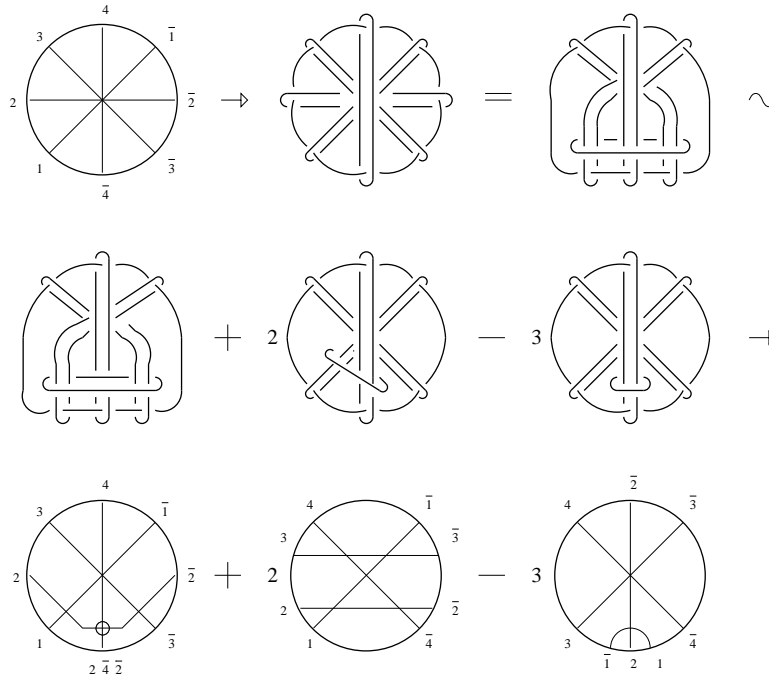
FIGURE 4.4. The proof of Theorem 1.1 when $k = 5$ as well as Theorem 1.2.

following chord diagram C with 4 chords. (Figure 4.4) Its word is given by

$$1234\bar{1}\bar{2}\bar{3}\bar{4} = [1, 234][2, 34][3, 4] \sim [1, 2][1, 3][1, 4][2, 3][2, 4][3, 4].$$

Using Lemma 3.1, this evaluates to three times the switch. On the other hand, if, instead of the outer loop, we choose to apply Lemma 3.1 to one of the loops associated with a chord, we will arrive at the following diagrammatic expansion. See Figure 4.4. We first change the position of the loop associated with the chord marked by 2 so that it is in a position for us to apply Lemma 3.1. We get the next linear combination of ASL's by applying Lemma 3.1. Notice the last two ASL's in this linear combination are associated with, respectively, chord diagrams $1234\bar{1}\bar{3}\bar{2}\bar{4}$ and $12\bar{1}34\bar{2}\bar{3}\bar{4}$. They can easily be evaluated to 2 times the switch and the switch respectively. The first ASL, though, is not associated with any chord diagram. But it still has the property that when we drop the outer circle, all the other components fall apart. Thus, the same method we used to evaluate a chord diagram can be used to evaluate this ASL. We use the first diagram in the last row in Figure 4.4 to denote this ASL. The small circle at the intersection of chord 2 and chord 4 indicates that when we want to recover the original ASL, the band associated with chord 2 runs through the band associated with chord 4. Hence we may read off the word represented by the outer circle as

$$1234\bar{1}\bar{2}\bar{3}2\bar{4}\bar{2} = [1, 234][2, 34][3, 4][4, 2]$$

and it evaluates to 4 times the switch. Thus we get the equality that 3 times the switch is equal to 5 times the switch. This is possible only when any $\mathbb{Z}$-homology 3-sphere invariant of order ≤ 5 evaluates zero on the switch. We therefore have proved Theorem 1.1 in the case when $k = 5$ as well as Theorem 1.2. $\qquad\square$

The proof of Theorem 1.1 in the case when $k > 5$ suggests that knots are too simple to hold up all finite type $\mathbb{Z}$-homology 3-sphere invariants. On the other hand, Lemma 4.6 and Theorem 1.2 tell us that a detailed knowledge about induced link invariants is of great help in understanding the original 3-manifold invariants. A natural question will be: What properties characterizes the induced invariant on ASL's with two components?

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